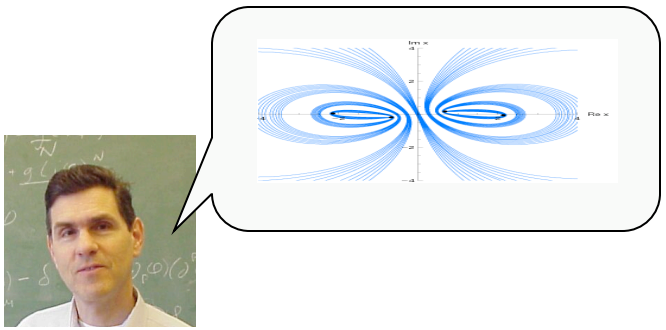
An Adaptive Packed-Memory Array

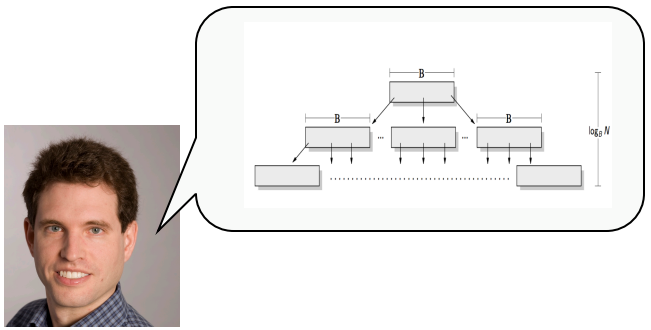
Michael A. Bender

Stony Brook and Tokutek^{R}, Inc

Physicists Beget Computer Scientists (Generational Divide)

- I'm a theoretical computer scientist
- Linguistic divide: Computer scientists speak discretely and physicists speak in quanta (and they speak continuously).





- But I want my talk in the spirit of QMCD '09.

My father has written papers …

with me:

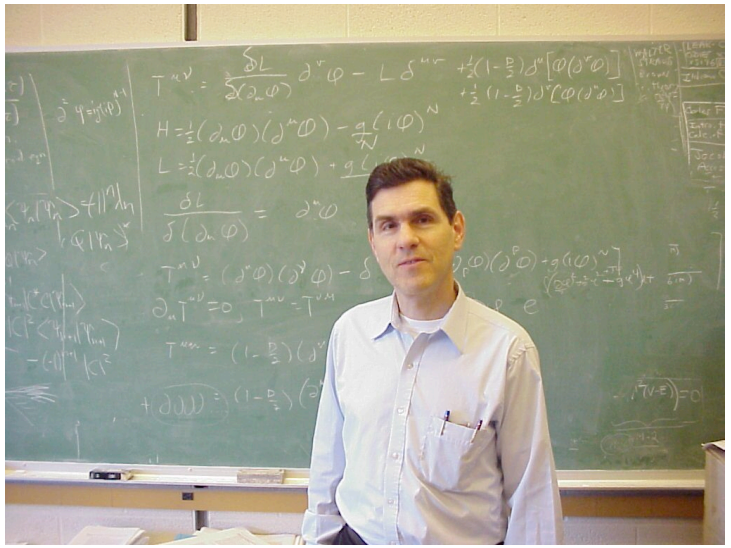
and my brother:

Oblong geyser

…but I won't talk about these.

(How I View)
My Father's Style

- Ask simple-to-state questions about elementary physics.
- Extract all the richness and complexity out of these problems
  - e.g., anharmonic oscillator, relativistic brachistochrone, QMCD



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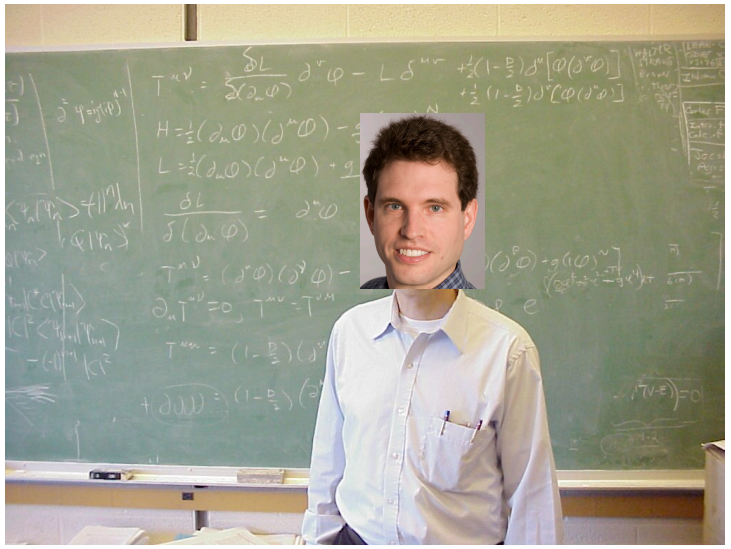
My Father's Style

- Ask simple-to-state questions about elementary physics.
- Extract all the richness and complexity out of these problems
  - e.g., anharmonic oscillator, relativistic brachistochrone, QMCD

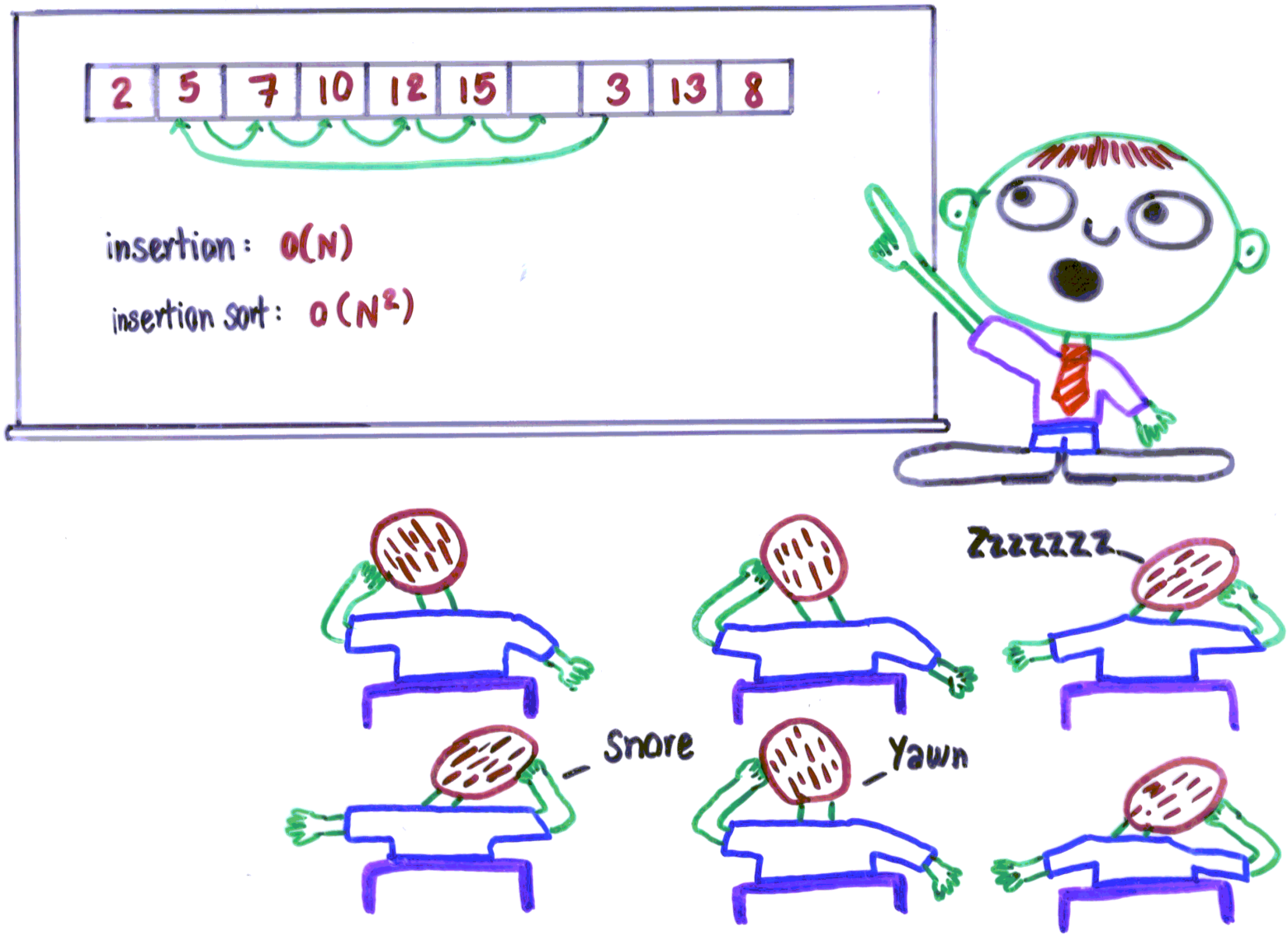
I've done my best to imitate this style

- Style that I strive for.
- Here's my attemPT to do the same sort of thing.



Fall 1990. I'm an undergraduate.

Insertion-Sort Lecture.



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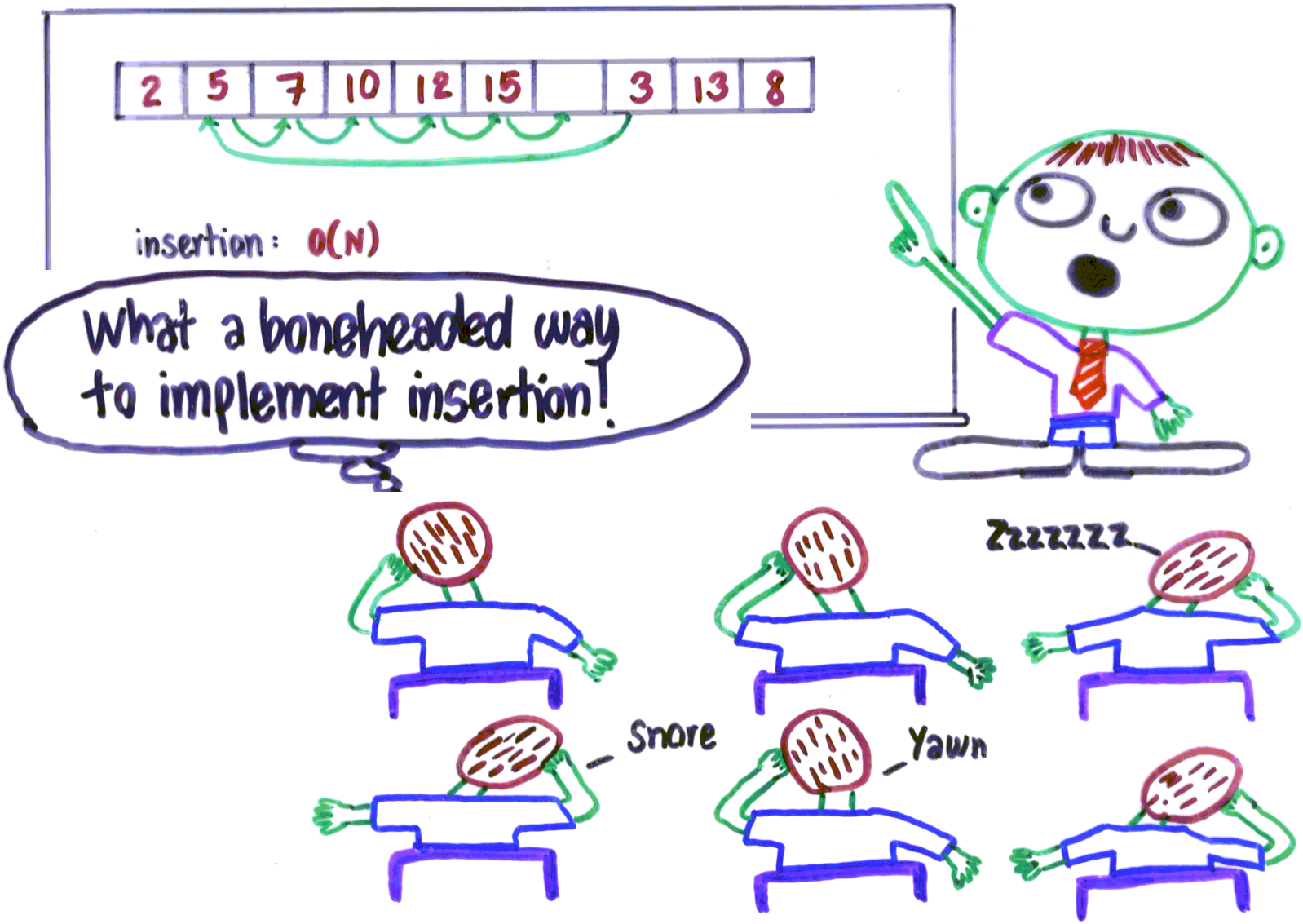

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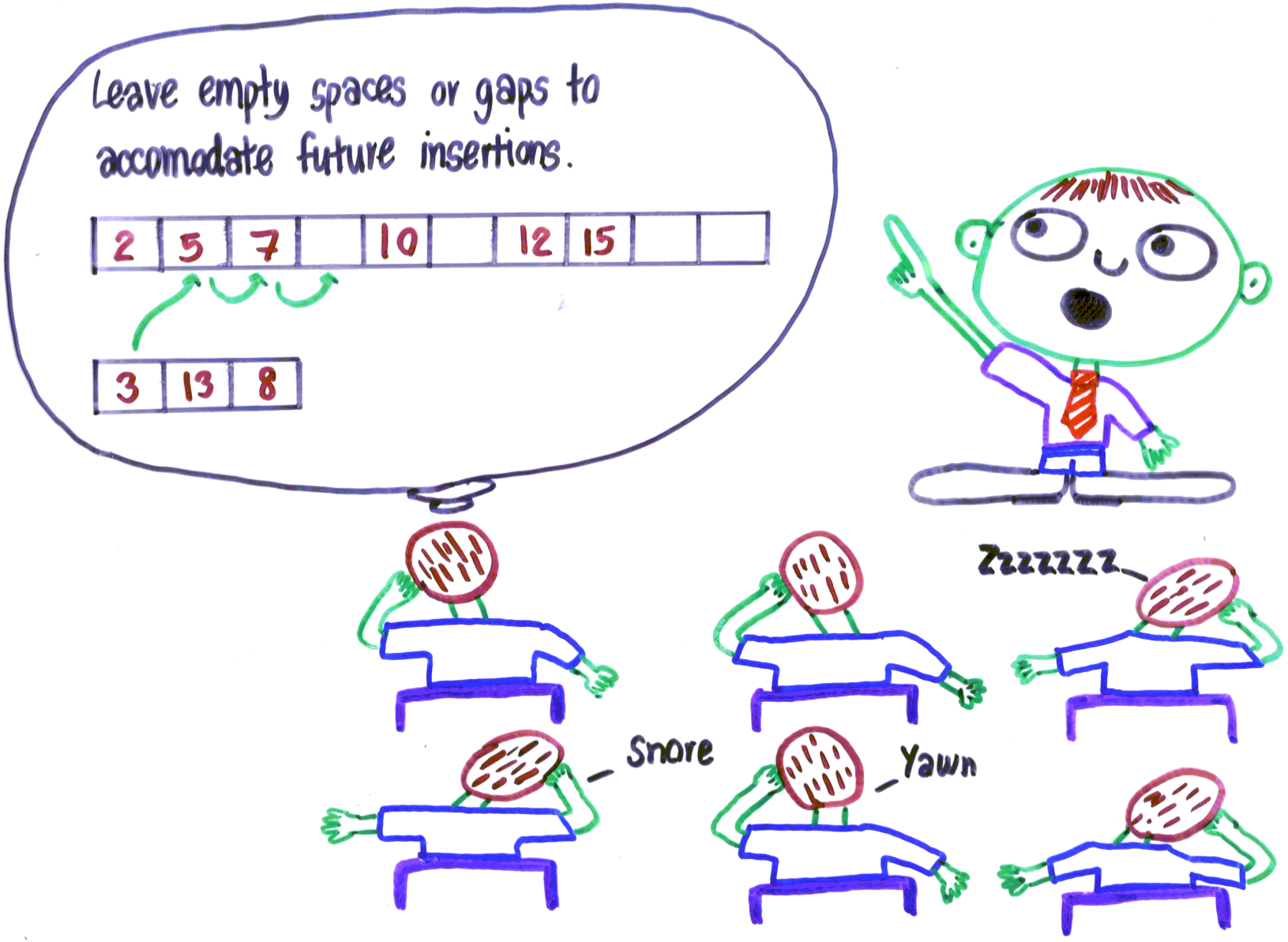
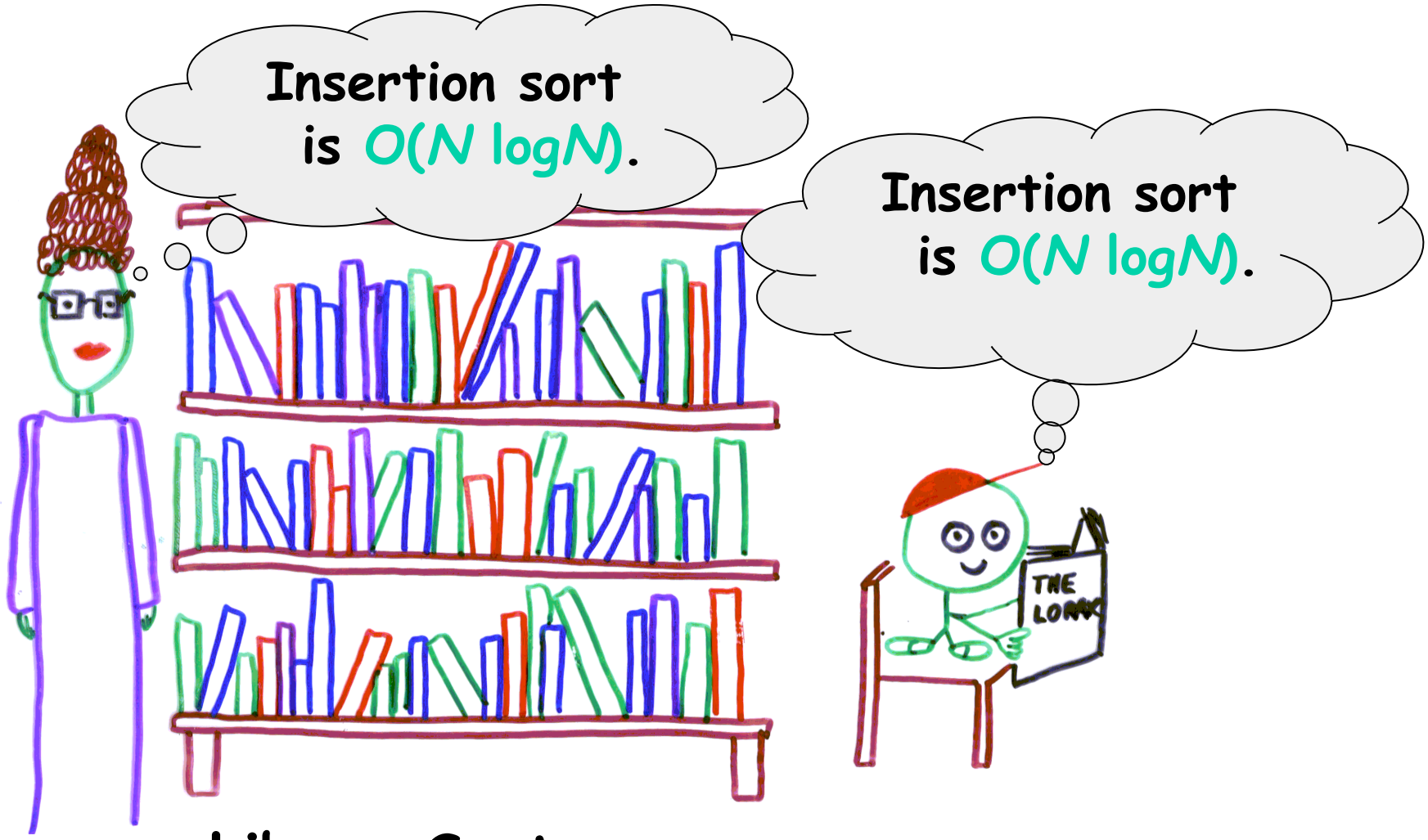


Anybody who has spent time in a library knows that insertions are cheaper than linear time.



LibrarySort [Bender,Farach-Colton,Mosteiro 04] :

$O(N \log N)$ sorting for average-case insertions.

How is LibrarySort like a library?

- Leave gaps on shelves so shelving is fast
- Putting books *randomly* on shelves with gaps: At most $O(\log N)$ books need to be moved with high probability* to make room for a new book.

* Probability $>1-1/\text{poly}(N)$, N=#books.

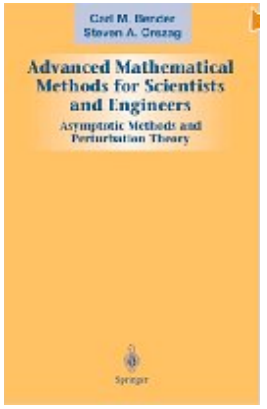
But what if Library buys many copies of….

- Bender and Orszag

But what if Library buys many copies of…?

- Bender and Orszag
- Bender and Orszag, 2nd Ed.
- Bender, and Orszag, 3rd Ed.

  ⋮
- Bender and Orszag, 500th Ed.



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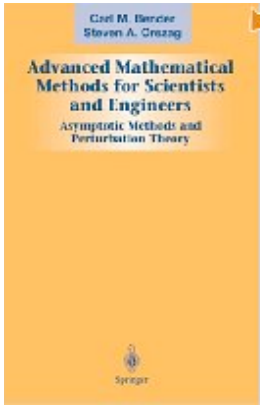
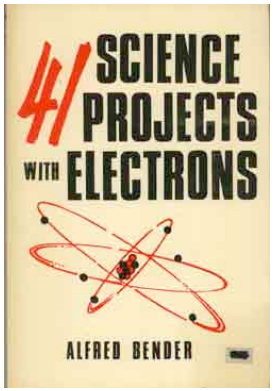
- Books by other Benders





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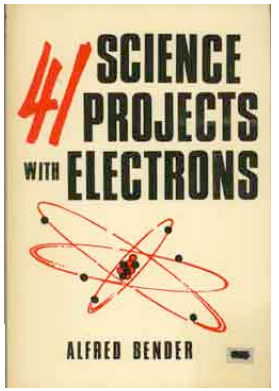

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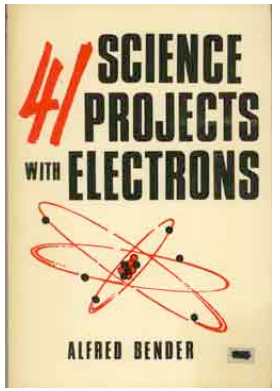
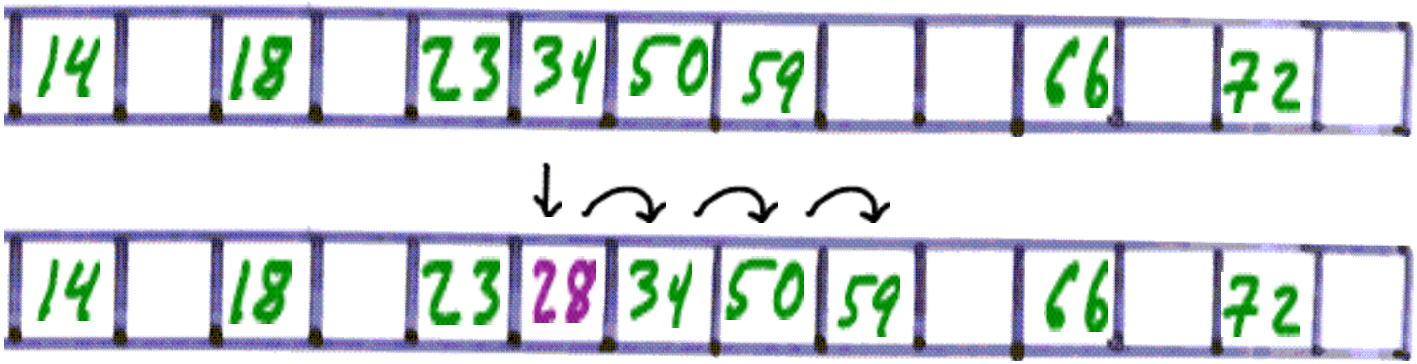


Now there's a bolus of books on in one place.
Can one still maintain small shelving costs?

…

Insertions into Array with Gaps

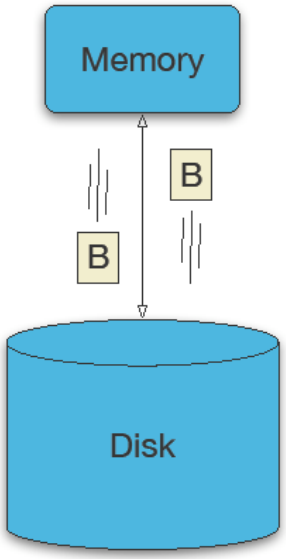
- Dynamically maintain N elements sorted in memory/on disk in a $\Theta(N)$ -sized array
  - Idea: rearrange elements & gaps to accommodate future insertions



- **Objective**: Minimize amortized (technical form of ave) # of elts moved per update.

Actually Two Objectives

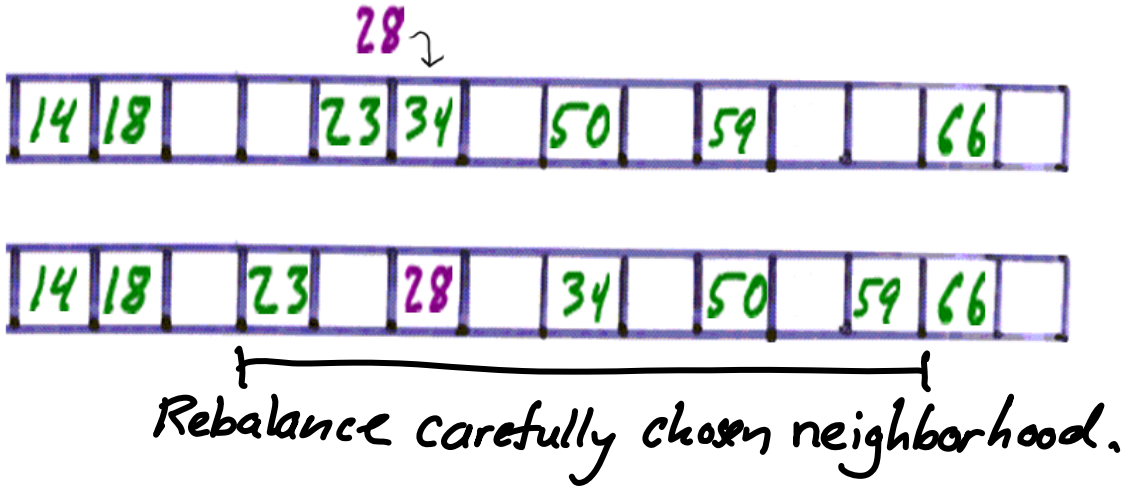
- Minimize # **elements moved** per insert.
- Minimize # **block transfers** per insert.
- Disk Access Model (DAM) of Computer
  - Two levels of memory
  - Two parameters:
    block size *B*, memory size *M*.



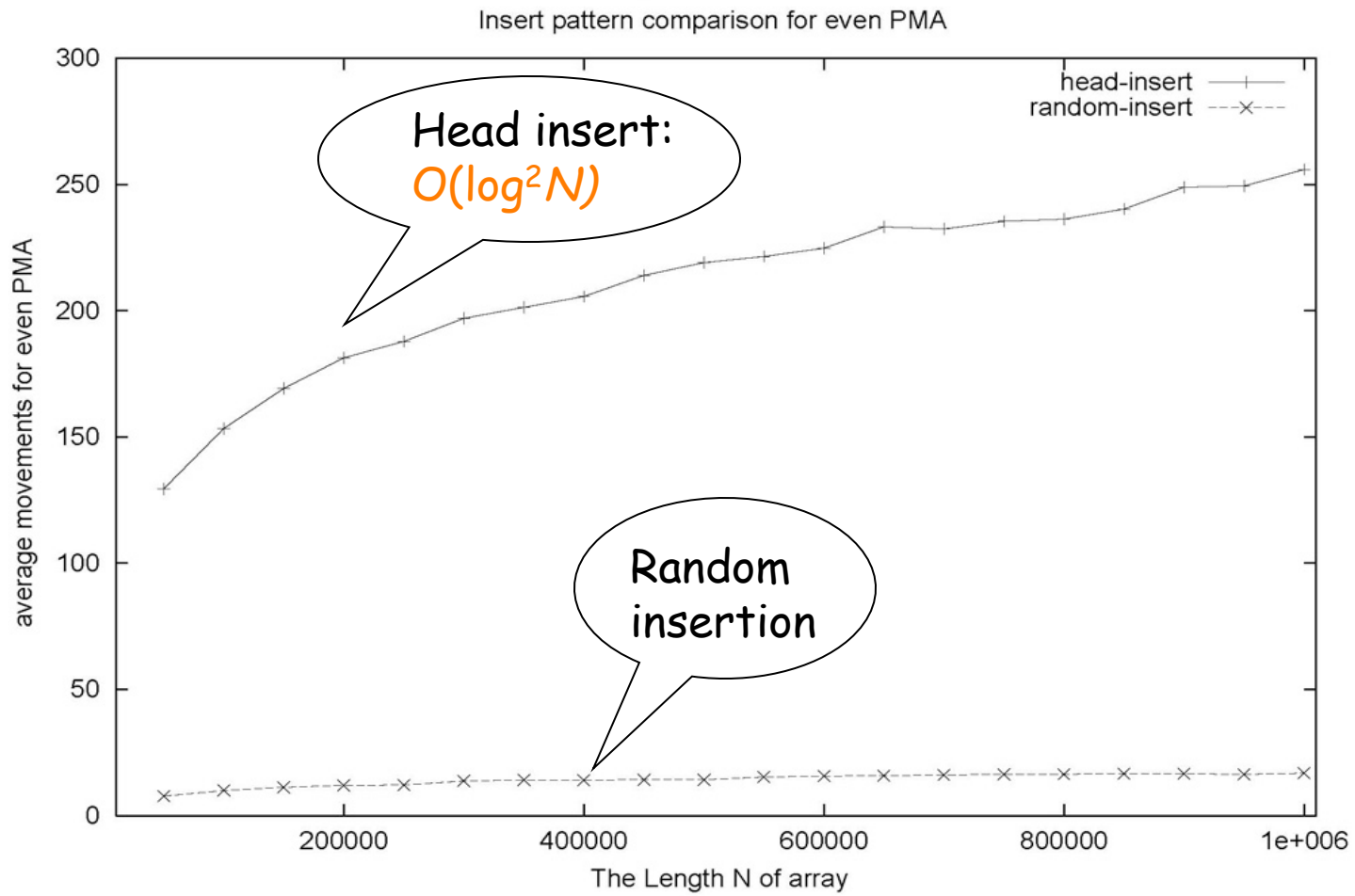
Packed-Memory Array (PMA)

[Bender, Demaine, Farach-Colton 00,05]

- **(Worst-case) Inserts/Deletes:**
  - $O(\log^2 N)$ amortized element moves
  - $O(1+(\log^2 N)/B)$ amortized memory transfers
- **Scans of *k* elements after given element:**
  - $O(1+k/B)$ memory transfers



PMA is good, but…



Problem: a worst case for PMA is sequential inserts, but this is a common case for databases. Industrial data structures (Oracle, TokuDB) are optimized for sequential inserts.

An Adaptive PMA

[Bender, Hu 2007]

- **Same** guarantees as PMA:

  $O(\log^2 N)$ element moves per insert/delete

  $O(1+(\log^2 N)/B)$ memory transfers

- Optimized for common insertion patterns:

  insert-at-head (sequential inserts)

  random inserts

  bulk inserts (repeatedly insert $O(N^b)$ elements in random position, $0 \le b \le 1$)

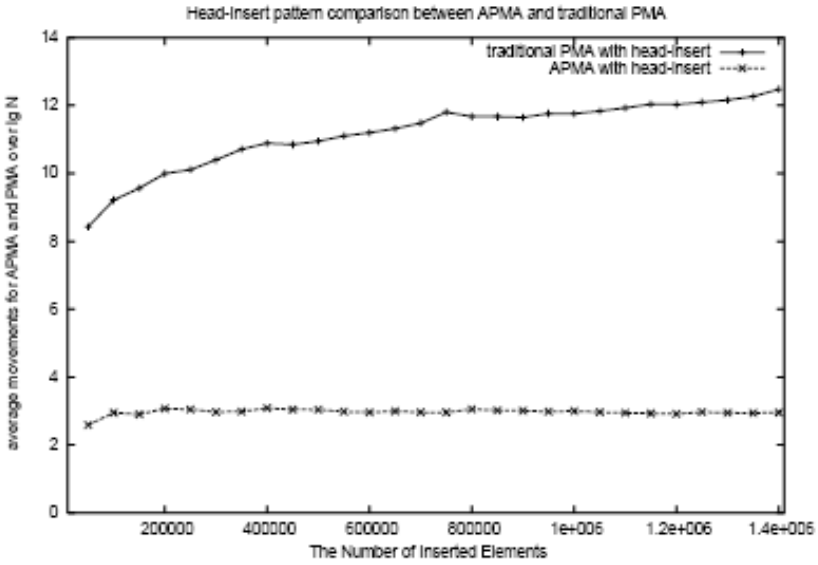
  Guarantees:

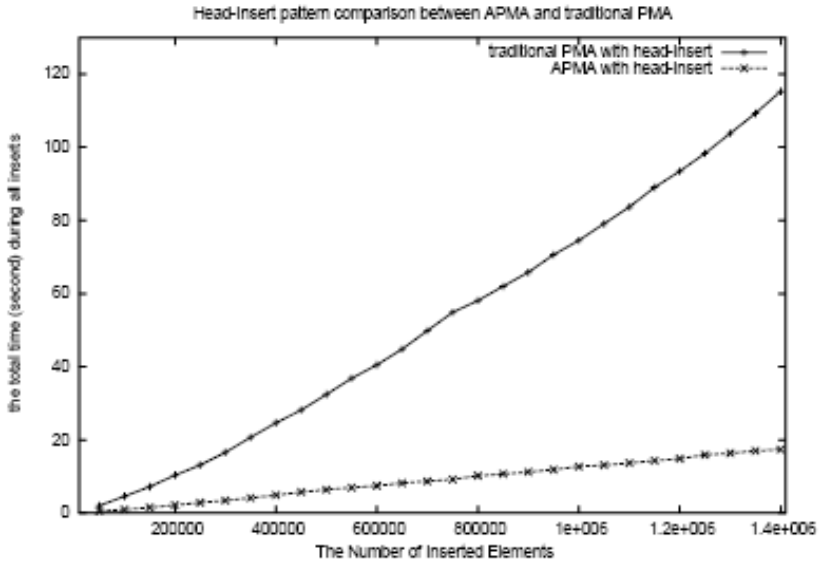
  $O(\log N)$ element moves

  $O(1+(\log N)/B)$ mem transfers

Sequential Inserts

Inserts "hammer" on one part of the array.



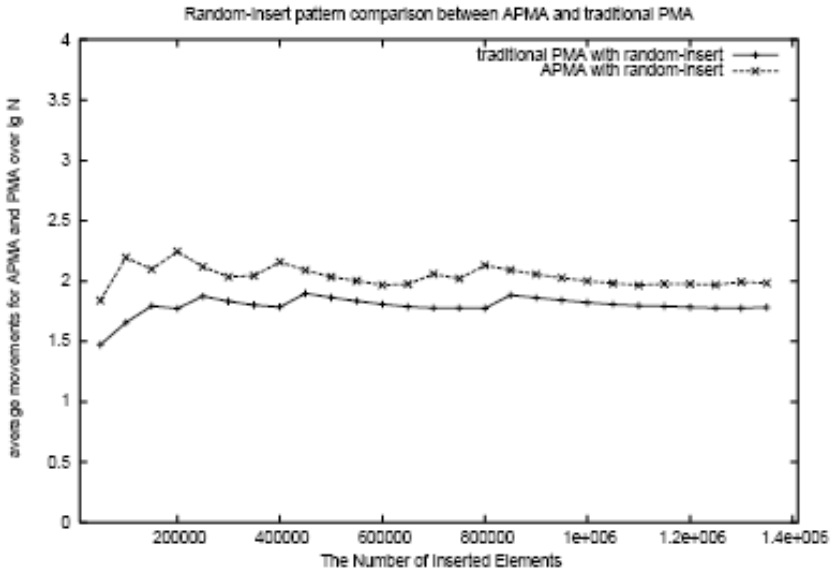


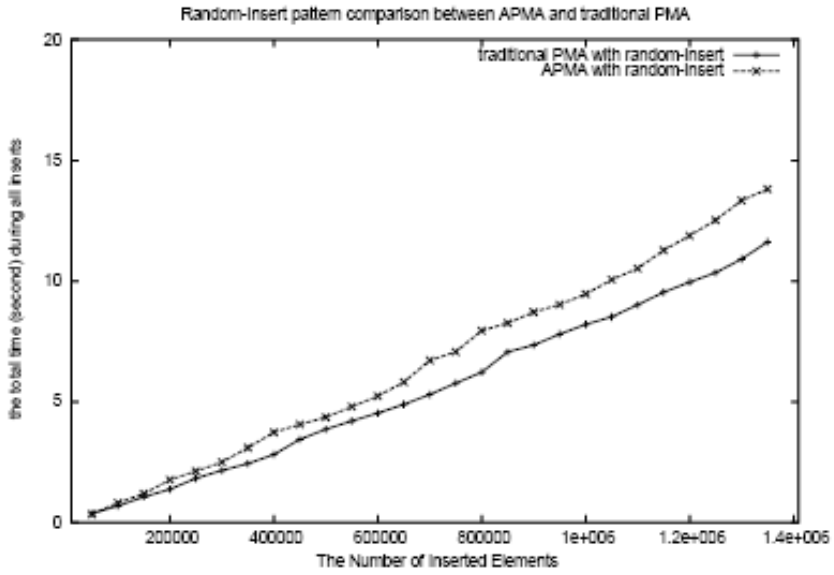
Amortized moves over $\log N$

running time

Random Inserts

Insertions are after random elements.





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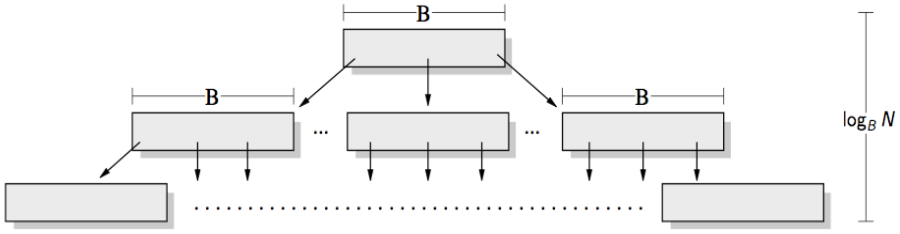
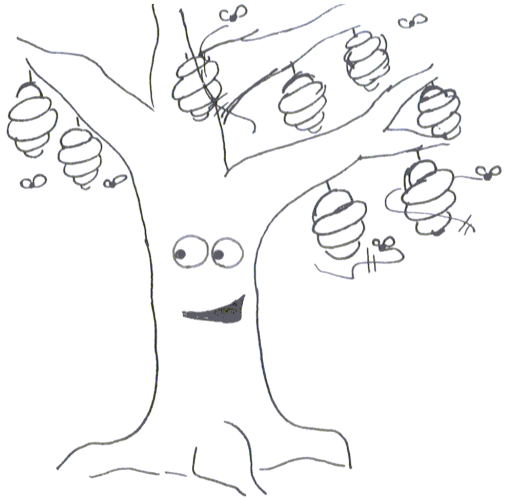
Sample Applications

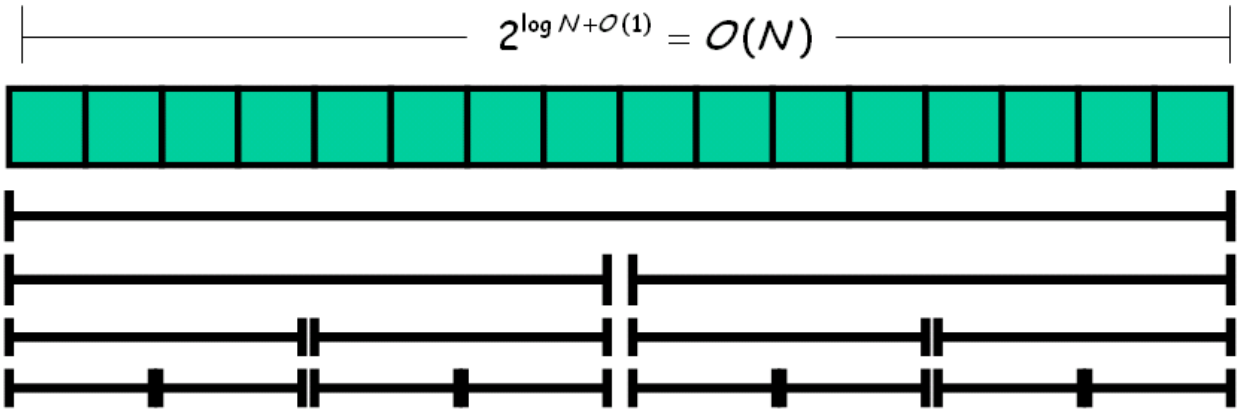




Maintain data physically in order on disk

- Traditional and "cache-oblivious" B-trees
  - Core of all databases and file systems
- My startup Tokutek
- Even an online dating website

Imaginary Intervals in PMA

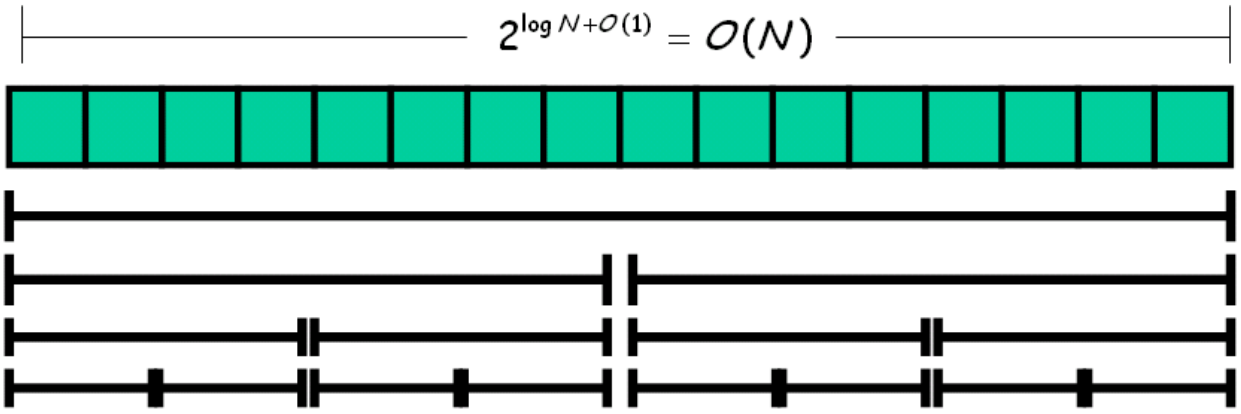


Density Thresholds

Upper	Lower
$\tau_3 = .7$	$\rho_3 = .3$
$\tau_2 = .8$	$\rho_2 = .25$
$\tau_1 = .9$	$\rho_1 = .2$
$\tau_0 = 1.0$	$\rho_0 = .15$

$$\tau_i - \tau_{i+1} = \Theta(\rho_{i+1} - \rho_i) = \Theta(1/\log N).$$

Imaginary Intervals in PMA



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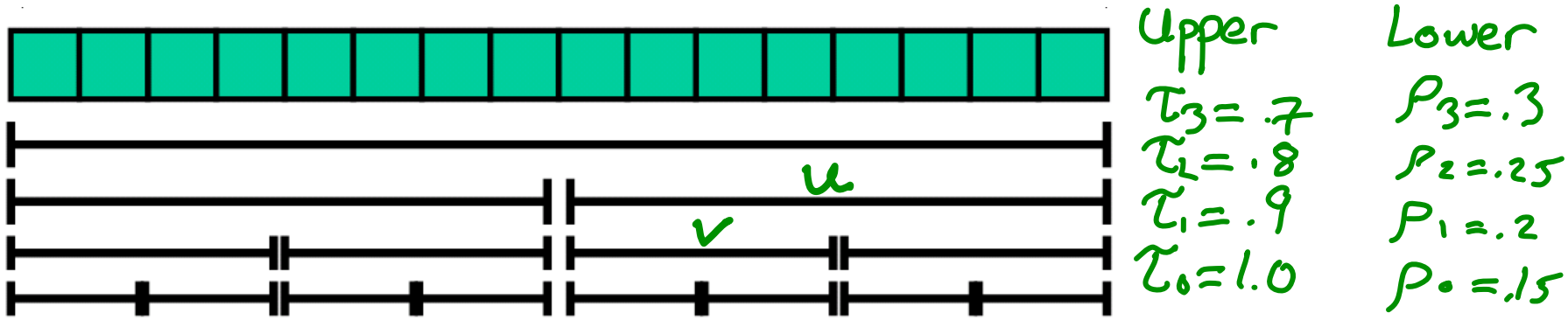
$$\tau_i - \tau_{i+1} = \Theta(\rho_{i+1} - \rho_i) = \Theta(1/\log N).$$

To insert:

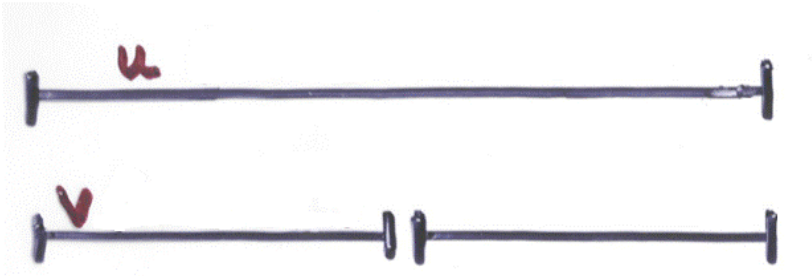
- Try to insert in leaf interval.
- If interval full, ***rebalance*** smallest enclosing interval within thresholds.

Analysis Idea: $O(\log^2 N)$ amortized element moves per insert

- $O(\log N)$ amort. moves to insert into interval
  - Amortized analysis: Charge rebalance of interval u to inserts into child interval v
- Insert in $O(\log N)$ intervals for insert in PMA



Analysis of $O(\log^2 N)$ Moves/Insert



density thresholds $\tau_{\ell+1}$, $\rho_{\ell+1}$

density thresholds τ_ℓ, ρ_ℓ

Before rebalance: $\text{density}(v) > \tau_\ell$ or $\text{density}(v) < \rho_\ell$

After rebalance: $\rho_{\ell+1} < \text{density}(v) < \tau_{\ell+1}$

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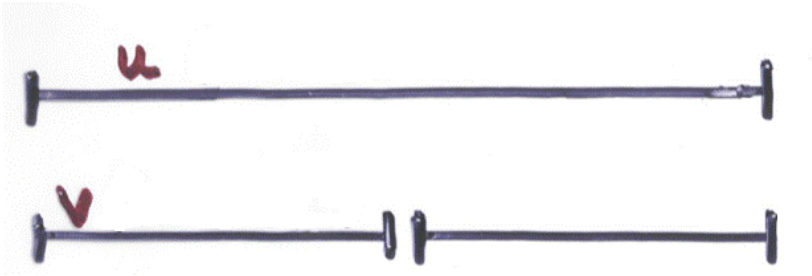


density thresholds $\tau_{\ell+1}$, $\rho_{\ell+1}$
density thresholds τ_ℓ, ρ_ℓ

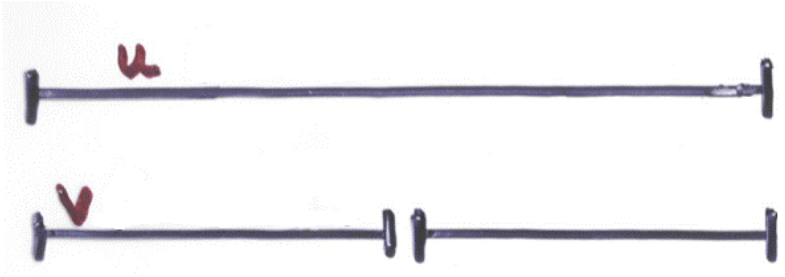
Before rebalance: $\text{density}(v) > \tau_\ell$ or $\text{density}(v) < \rho_\ell$

After rebalance: $\rho_{\ell+1} < \text{density}(v) < \tau_{\ell+1}$

$$\text{Amortized cost of rebalancing } u = \frac{\text{cost of rebalance}}{\text{inserts betw. rebalance}} = \frac{\text{size}(u)}{\text{size}(v)} \text{Max}\left\{\frac{1}{\tau_\ell - \tau_{\ell+1}}, \frac{1}{\rho_{\ell+1} - \rho_\ell}\right\}$$

$$= O(\log N)$$

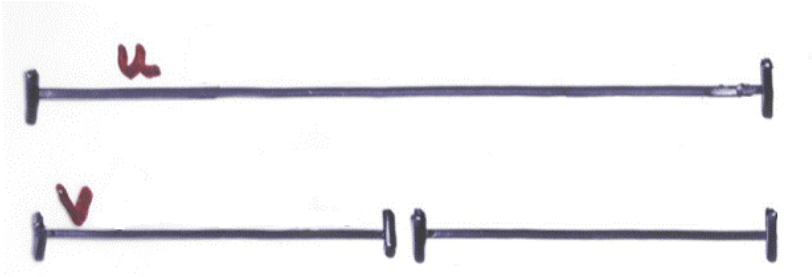
Analysis Summary



density thresholds $\tau_{\ell+1}$, $\rho_{\ell+1}$
density thresholds τ_ℓ , ρ_ℓ

- Charge rebalance cost of u to inserts into v
  - After rebalance v within threshold of parent u
- Amortized cost of $O(\log N)$ to insert into u

Analysis Summary



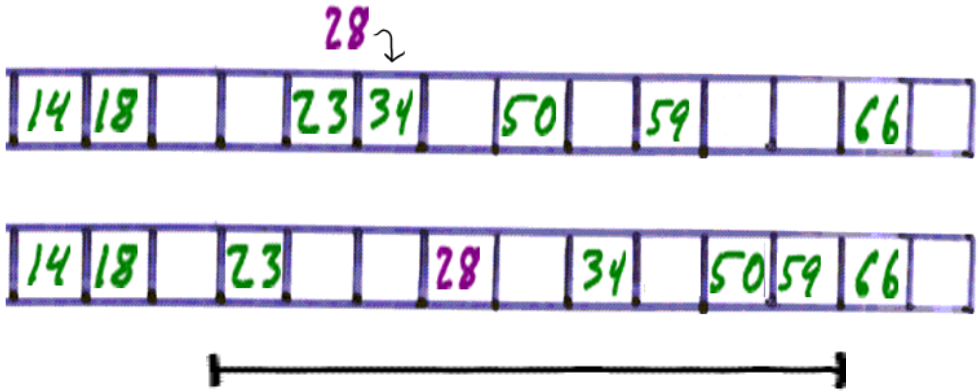
density thresholds $\tau_{\ell+1}$, $\rho_{\ell+1}$
density thresholds τ_ℓ , ρ_ℓ

- Charge rebalance cost of u to inserts into v
  - After rebalance v within threshold of parent u
- Amortized cost of $O(\log N)$ to insert into u
- But each insert is into $O(\log N)$ intervals

- Total: $O(\log^2 N)$ amortized moves

Idea of Adaptive PMA

- Adaptively remember elements that have many recent inserts nearby.
- Rebalance *unevenly*.
  Add extra space near these volatile elements.

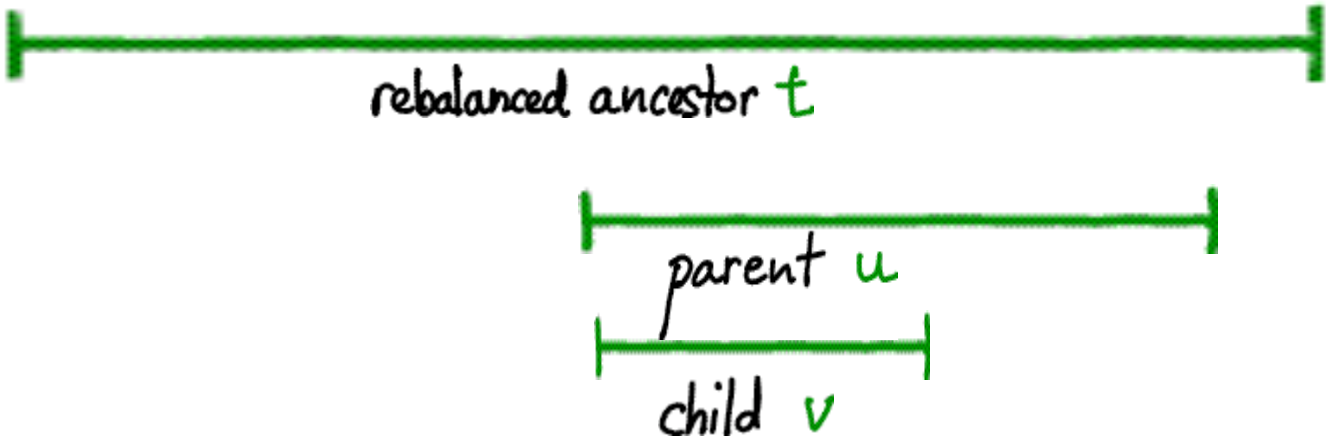


- This strategy overcomes a $\Omega(\log^2 N)$ lower bound [Dietz, Sieferas, Zhang 94] for "smooth" rebalances

Why $O(\log^2 N)$ Can Be Improved in the Common Case.

To guarantee $O(\log^2 N)$, we only need….

Rebalance Property: After a rebalance involving v, v is within parent u's density threshold.

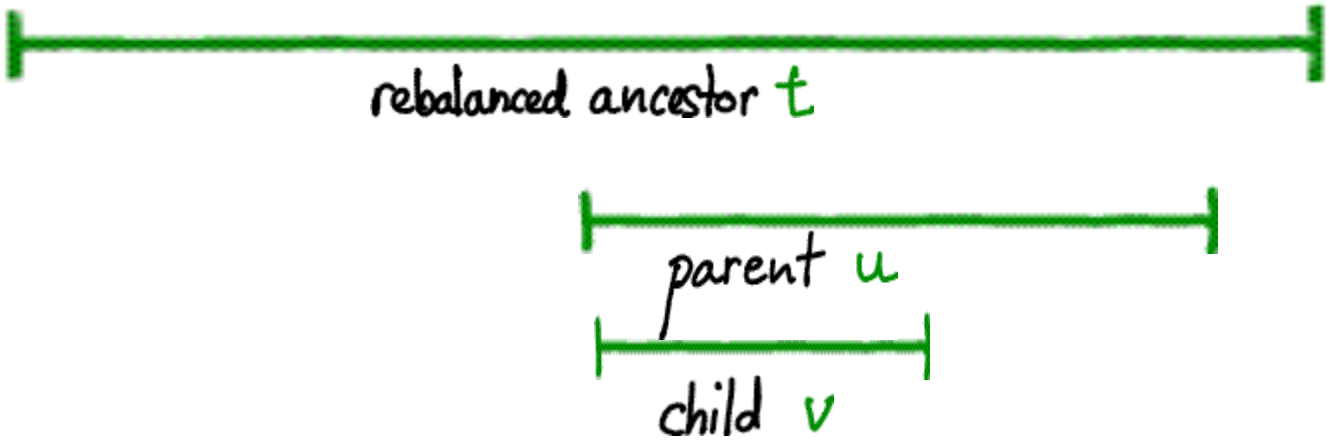


Summary: As long as v is within u's threshold, it can be sparser or denser than t's density thresholds.

Why $O(\log^2 N)$ Can Be Improved in the Common Case.

To guarantee $O(\log^2 N)$, we only need….

Rebalance Property: After a rebalance involving v, v is within parent u's density threshold.

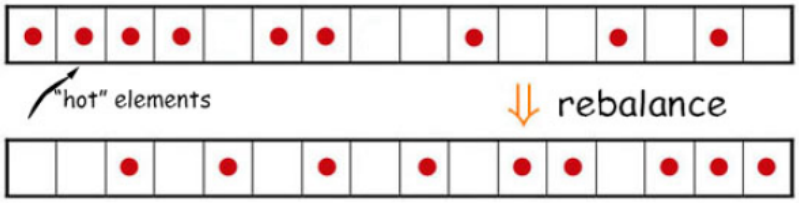


Summary: Remarkable that this is good enough.
Only large rebalances have slop and most are small.

How to Remember Hot Elements Adaptively

- Maintain an $O(\log N)$-sized **predictor**, which keeps track of PMA regions with recent inserts
  - $O(\log N)$ counters, each up to $O(\log N)$.
  - Remembers up to $O(\log N)$ hotspot elements.
  - Tolerates "random noise" in inputs.

- (Generalization of how to find majority element in an array with a single counter.)



- Rebalance to even out weight of counters, while maintaining rebalance property.

Summary

- Insertion sort with gaps
  - LibrarySort [Bender,Farach,Colton,Mosteiro '04] (+ Wikipedia entry)
- Worst-possible inserts
  - PMA [Bender,Demaine,Farach-Colton '00,'05]
  - Cache-oblivious B-trees and other data structures
- Adapt to common distributions
  - APMA [Bender,Demaine,Farach-Colton '00,'05]
- Implementation of cache-oblivious data structures
  - Tokutek

- Is it practical to keep data physically in order in memory/on disk?

Overview of Talk